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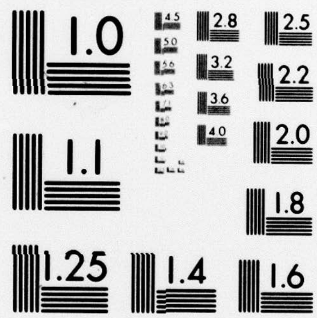
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Hedy Attouch

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ON THE MAXIMALITY OF THE SUM OF TWO MAXIMAL MONOTONE OPERATORS

Hedy Attouch

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ABSTRACT

We prove that the topological condition $\mathrm{Int}(D(A) - D(B)) \ni 0$, implies that the sum of the two maximal monotone operators A and B is still maximal monotone.

AMS (MOS) Subject Classifications: 47H05, 49A40

Key Words: Maximal monotone operators, Duality methods in optimization, Convexity

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SIGNIFICANCE AND EXPLANATION

A wide variety of problems involving nonlinear partial differential equations, subject to boundary conditions, may be shown to have solutions by establishing that the associated differential operators satisfy a certain technical condition.

This condition, called maximal monotonicity, allows the use of a well developed abstract theory which includes results about existence, regularity, etc., of solutions of operator equations.

It is quite useful, therefore, to have easily verified conditions which imply that an operator is maximal monotone.

Frequently an operator may be regarded as the sum of simpler components. This paper gives a new sufficient condition which guarantees that the sum of two maximal monotone operators is again maximal monotone.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE MAXIMALITY OF THE SUM OF TWO MAXIMAL MONOTONE OPERATORS[†]

Hedy Attouch

Introduction

Let A and B be two maximal monotone operators in a real Hilbert space H. The classical theorem of R. T. Rockafellar [7], H. Brezis [2] tells us that if $D(A) \cap \mathrm{Int}\, D(B) \neq \phi$, then the sum $A + B$ is still maximal monotone. We show that the same conclusion holds under the weaker, symmetric assumption: $\mathrm{Int}(D(A) - D(B)) \ni 0$ (Theorem 1)[*]. When one of the operators is a subdifferential, $A = \partial\varphi$, (resp. when the two operators are subdifferentials $A = \partial\varphi$, $B = \partial\psi$) this condition can be weakened to $\mathrm{Int}(D(\varphi) - D(B)) \ni 0$ (Theorem 2) (resp. $\mathrm{Int}(D(\varphi) - D(\psi)) \ni 0$ (Theorem 3)). These results are intimately related (Remark 2 and Remark 3) to:

a) H. Brezis and A. Haraux in [4] give sufficient conditions on two maximal monotone operators A and B, in order that $R(A+B) \simeq R(A) + R(B)$ (i.e. that the range of the sum and the sum of the ranges have same interior and closure).

b) The classical "Slater stability condition" (Continuity of a functional at one point of the domain of the other) can be weakened (cf. for example, J. P. Aubin [1]) to the following symmetric one: the difference of their domains is a neighbourhood of the origin.

I thank very much J. P. Aubin and H. Brezis for their advise and stimulating discussions.

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* In fact (cf. Remark 2) one can weaken this assumption to $\mathrm{Int}(\mathrm{Conv}\, D(A) - \mathrm{Conv}\, D(B)) \ni 0$.

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Let H be a real Hilbert space with the norm $|\cdot|$, and the scalar product $\langle\cdot,\cdot\rangle$. Given A a maximal monotone operator in H, we shall denote by $D(A)$ its domain, by $R(A)$ its range, by $A_\lambda = \frac{1}{\lambda}(I - (I+\lambda A)^{-1})$, for $\lambda > 0$, its Yosida approximation and by A^0 its minimal section.

<u>Theorem 1</u>

Let A and B two maximal monotone operators in H.

Let us assume that $0 \in \mathrm{Int}(D(A) - D(B))$.(*)

Then $A + B$ is a maximal monotone operator.

<u>Proof of Theorem 1</u>

Given f in H and $\lambda > 0$, let us consider u_λ the solution of

$$(1_\lambda) \qquad u_\lambda + Au_\lambda + B_\lambda u_\lambda \ni f$$

and let us prove that $\sup_{\lambda>0} |B_\lambda u_\lambda| < +\infty$; from the Brezis/Crandall/Pazy theorem [3] it will follow that the u_λ converges, as λ goes to zero, to a solution u of

$$(1) \qquad u + Au + Bu \ni f \quad ,$$

and this implies the maximality of $A + B$.

We first observe that the $(u_\lambda)\lambda > 0$, are bounded in H: taking $x \in D(A) \cap D(B)$, and using the monotonicity of $A + B_\lambda$, one has $|u_\lambda - x| \leq |f - (x + A^0x + B_\lambda x)|$ which implies $|u_\lambda| \leq 2|x| + |f| + |A^0x| + |B^0x|$.

By assumption, there exists a $\rho > 0$ such that

$$\forall y \in H, |y| < \rho \qquad y \in D(A) - D(B) \quad .$$

Given such a y, it suffices to show that $\sup_{\lambda>0} \langle B_\lambda u_\lambda, y\rangle \leq C(y) < +\infty$; the conclusion will then follow from the uniform boundedness theorem.

If $y \in H$, y can be written as $y = \beta - \alpha$ with $\beta \in D(B)$, $\alpha \in D(A)$, and

(*) That is to say, the algebraic difference of their domains is a neighbourhood of the origin.

$$\langle B_\lambda u_\lambda, y\rangle = \langle B_\lambda u_\lambda, \beta\rangle - \langle B_\lambda u_\lambda, \alpha\rangle \ .$$

a) From the monotonicity of B_λ,

$$\langle B_\lambda u_\lambda - B_\lambda \beta,\ u_\lambda - \beta\rangle \geq 0, \quad \text{which implies}$$

$$\langle B_\lambda u_\lambda, \beta\rangle \leq \langle B_\lambda u_\lambda, u_\lambda\rangle - \langle B_\lambda \beta, u_\lambda - \beta\rangle \ ;$$

$$\text{so,}\ \langle B_\lambda u_\lambda, y\rangle \leq \langle B_\lambda u_\lambda, u_\lambda - \alpha\rangle + |B^0\beta|\cdot|u_\lambda - \beta| \ .$$

b) Since u_λ satisfies (1_λ), $B_\lambda u_\lambda = f - u_\lambda - v_\lambda$ with $v_\lambda \in Au_\lambda$, and the preceding inequality becomes:

$$\langle B_\lambda u_\lambda, y\rangle \leq \langle f - u_\lambda - v_\lambda,\ u_\lambda - \alpha\rangle + |B^0\beta|\cdot|u_\lambda - \beta| \ .$$

c) From the monotonicity of A,

$$\langle v_\lambda - A^0\alpha,\ u_\lambda - \alpha\rangle \geq 0 \Rightarrow - \langle v_\lambda,\ u_\lambda - \alpha\rangle \leq |A^0\alpha|\cdot|u_\lambda - \alpha| \ .$$

Finally,

$$\langle B_\lambda u_\lambda, y\rangle \leq |u_\lambda - \alpha|[|f - u_\lambda| + |A^0\alpha|] + |u_\lambda - \beta|\cdot|B^0\beta| \leq C(y)$$

since the (u_λ) are bounded for $\lambda > 0$.

Remark 1. The condition $\mathrm{Int}(D(A) - D(B)) \ni 0$, is symmetric in A and B and is weaker than the classical condition of Rockafellar [7], H. Brezis [2], $\mathrm{Int}\, D(A) \cap D(B) \neq \phi$, since $\mathrm{Int}(D(A) - D(B)) \supset \mathrm{Int}\, D(A) - D(B)$. Clearly this weaker condition can be satisfied even if $D(A)$ and $D(B)$ have an empty interior domain!

Let us see now how to improve this result when one of the operator is a subdifferential.

Theorem 2

Let B be a maximal monotone operator in H and $\partial\varphi$ be the subdifferential of φ, a convex, lower semi continuous, proper function from H into $]-\infty,+\infty]$.

Let us assume that $0 \in \mathrm{Int}(D(\varphi) - D(B))$

Then $\partial\varphi + B$ is a maximal monotone operator.

Proof of Theorem 2

As in the proof of Theorem 1, we consider the solution u_λ of

$$(2_\lambda) \qquad u_\lambda + \partial\varphi(u_\lambda) + B_\lambda u_\lambda \ni f \quad .$$

We want to prove that $\sup_\lambda |B_\lambda u_\lambda| < +\infty$. The equation (2_λ) can be written

$$(2_\lambda)_{bis} \qquad \forall v \in H \quad 0 \geq \varphi(u_\lambda) - \varphi(v) + \langle f - u_\lambda - B_\lambda u_\lambda, v - u_\lambda \rangle \quad .$$

The $(u_\lambda)_{\lambda>0}$ are bounded in H: Let us take $v_0 \in D(\varphi) \cap D(B)$ and $b, c \in \mathbb{R}^+$ such that $\varphi(x) + b|x| + c \geq 0$; Since $\langle B_\lambda u_\lambda - B_\lambda v_0, u_\lambda - v_0 \rangle \geq 0$, from $(2_\lambda)_{bis}$ we get:

$$-b|u_\lambda| - c - \varphi(v_0) + \langle f - u_\lambda, v_0 - u_\lambda \rangle + \langle B_\lambda v_0, u_\lambda - v_0 \rangle \leq 0 \quad ;$$

so there exist p_1 and $p_2 \in \mathbb{R}$ such that: $\forall \lambda > 0 \quad |u_\lambda|^2 + p_1|u_\lambda| + p_2 \leq 0$, which implies that the (u_λ) are bounded for $\lambda > 0$.

By assumption, there exists a $\rho > 0$ such that

$$\forall y \in H \quad |y| < \rho \qquad y \in D(\varphi) - D(B) \quad .$$

Let us prove that for any such y

$$\sup_\lambda \langle B_\lambda u_\lambda, y \rangle \leq C(y) < +\infty \quad ,$$

the conclusion will follow from the uniform boundedness theorem; so $y = \beta - \alpha$, $\beta \in D(B)$, $\alpha \in D(\varphi)$ and

$$\langle B_\lambda u_\lambda, y \rangle = \langle B_\lambda u_\lambda, \beta \rangle - \langle B_\lambda u_\lambda, \alpha \rangle \quad .$$

From the monotonicity of B_λ

$$\langle B_\lambda u_\lambda, y \rangle \leq \langle B_\lambda u_\lambda, u_\lambda - \alpha \rangle + |B^0\beta| \cdot |u_\lambda - \beta| \quad .$$

Now take $v = \alpha$ in $(2_\lambda)_{bis}$:

$$\langle B_\lambda u_\lambda, u_\lambda - \alpha \rangle \leq \varphi(\alpha) - \varphi(u_\lambda) + \langle f - u_\lambda, u_\lambda - \alpha \rangle \quad .$$

Combining the two last inequalities, we get

$$\langle B_\lambda u_\lambda, y\rangle \leq \varphi(\alpha) + b|u_\lambda| + c + |f - u_\lambda|\cdot|u_\lambda - \alpha| + |B^0\beta|\cdot|u_\lambda - \beta|$$

$$\leq C(y) \quad \text{since the} \quad (u_\lambda)_{\lambda>0} \quad \text{are bounded.}$$

Finally, when the two operators are subdifferentials we obtain the following result: We denote by φ^* the conjugate function of φ and by ∇ the inf convolution (i.e. given two functions φ_1 and φ_2, then $\varphi_1 \nabla \varphi_2(x) = \inf_y\{\varphi_1(x-y) + \varphi_2(y)\}$).

Theorem 3

Let φ and ψ two convex, lower semi-continuous, proper functions from H into $]-\infty,+\infty]$.

Let us assume that $0 \in \mathrm{Int}(D(\varphi) - D(\psi))$.

Then $\varphi^* \nabla \psi^*$ is lower semi continuous, the inf-convolution is exact and $\partial\varphi + \partial\psi$ is a maximal monotone operator.

Proof of Theorem 3

a) Let $c \in \mathbb{R}$ and $f_n \xrightarrow{s-H} f$ such that $\varphi^* \nabla \psi^*(f_n) \leq c$; let us prove that $\varphi^* \nabla \psi^*(f) \leq c$; from the definition of the *inf-convolution, there exist* $\varepsilon_n \to 0$ and $u_n \in H$ such that:

$$\varphi^*(u_n) + \psi^*(f_n - u_n) \leq C + \varepsilon_n \quad . \tag{3}$$

Let us prove that $\sup_n |u_n| < +\infty$; then taking a weakly converging subsequence $u_{n_k} \xrightarrow{w-H} \bar{u}$, by the lower semi-continuity of φ^* and ψ^*, we shall get

$$\varphi^*(\bar{u}) + \psi^*(f - \bar{u}) \leq C \quad \text{and therefore} \quad \varphi^* \nabla \psi^*(f) \leq C \quad .$$

The same argument will tell us that

$$\forall f \in D(\varphi^* \nabla \psi^*) \ \exists u \in H \ \text{such that} \ \varphi^* \nabla \psi^*(f) = \varphi^*(u) + \psi^*(f - u)$$

(i.e. the inf-convolution is exact).

In order to prove the boundedness of the $(u_n)_{n\in\mathbb{N}}$ we apply the uniform boundedness theorem: let $\rho > 0$ such that

$$\forall y \in H \quad |y| < \rho \Rightarrow y \in D(\varphi) - D(\psi)$$

and let us prove that for any such y $\sup_n \langle u_n, y\rangle \leq C(y) < +\infty$.

So $y = \alpha-\beta$, with $\alpha \in D(\varphi)$, $\beta \in D(\psi)$ and

$$\langle u_n,y\rangle = \langle u_n,\alpha\rangle + \langle f - u_n,\beta\rangle - \langle f,\beta\rangle$$

$$\leq \varphi^*(u_n) + \varphi(\alpha) + \psi^*(f - u_n) + \psi(\beta) - \langle f,\beta\rangle \quad .$$

From (3)

$$\langle u_n,y\rangle \leq C + \varepsilon_n + \varphi(\alpha) + \psi(\beta) - \langle f,\beta\rangle$$

and the $(u_n)_{n \in \mathbb{N}}$ are bounded.

b) Let us take $(u,f) \in \partial(\varphi+\psi)$ and let us prove that $(u,f) \in \partial\varphi+\partial\psi$; from the maximal monotony of $\partial(\varphi+\psi)$ the result will follow:

$$(u,f) \in \partial(\varphi+\psi) \Longleftrightarrow (\varphi+\psi)(u) + (\varphi+\psi)^*(f) - \langle f,u\rangle = 0 \quad .$$

From a), $(\varphi+\psi)^* = (\varphi^* \nabla \psi^*)^{**} = \varphi^* \nabla \psi^*$, since $\varphi^* \nabla \psi^*$ is lower-semi-continuous. Moreover, there exists $\bar{u} \in H$ such that $(\varphi^* \nabla \psi^*)(f) = \varphi^*(\bar{u}) + \psi^*(f - \bar{u})$. So, $\{\varphi(u) + \varphi^*(\bar{u}) - \langle u,\bar{u}\rangle\} + \{\psi(u) + \psi^*(f - \bar{u}) - \langle u,f-\bar{u}\rangle\} = 0$. Since each of these quantities is positive, each is equal to zero which means that

$$\bar{u} \in \partial\varphi(u),\ f-\bar{u} \in \partial\psi(u) \quad \text{and} \quad f = \bar{u} + (f-\bar{u}) \in (\partial\varphi+\partial\psi)(u) \quad .$$

Remark 2

The Brezis-Haraux theorem [4] states:

Let A and B two maximal monotone operators satisfying:

$A+B$ is maximal monotone

A and B satisfy the condtion (*)

then $R(A+B) \simeq R(A) + R(B)$.

We recall that a monotone operator S satisfies (*) if

$$\forall x \in D(S)\ \forall y \in D(S)\ \exists C(x,y) \text{ such that: } \forall z \in D(A)\ \langle Sz-Sx,z-y\rangle \geq C(x,y) \quad .$$

As noticed by Brezis, one can deduce the theorem 1 from this theorem through the following transformation:

Let us consider the equation (1) $2u + Au + Bu \ni f$, which we want to solve.

Let us write it $u + Au + u + Bu \ni f$; since $I + B$ is onto, let us define $v = u + Bu$ as a new variable; the equation to be solved becomes

$$(I+B)^{-1}(v) - (I+A)^{-1}(f-v) = 0 \quad . \tag{4}$$

We apply the B/H theorem with $Sv = (I+B)^{-1}(v)$, $Tv = -(I+A)^{-1}(f-v)$. Clearly, these two operators are maximal monotone continuous operators and their sum is maximal monotone. They satisfy the condition (*):

$$\begin{aligned}\langle (I+B)^{-1}(z) - (I+B)^{-1}(x), z-y\rangle &= \langle u-v, z-y\rangle \quad \text{with} \quad u = (I+B)^{-1}z,\ v = (I+B)^{-1}x \\ &= \langle u-v, u + Bu - y\rangle \\ &= \langle u-v, u-v + (v+Bv) + (Bu-Bv) - y\rangle \\ &\geq |u-v|^2 - |x+y|\cdot|u-v| \geq C(x,y) \quad .\end{aligned}$$

Therefore, $R((I+B)^{-1} - (I+A)^{-1}(f-\ )) \supset \text{Int}[\text{Conv } R(I+B)^{-1} + \text{Conv } R(-(I+A)^{-1}(f-\ ))]$

$$\supset \text{Int } [\text{Conv } D(B) - \text{Conv } D(A)] \quad .$$

So if we assume that $\text{Int}[\text{Conv } D(B) - \text{Conv } D(A)] \ni 0$ we can solve (4) and therefore (1).

<u>Remark 3</u>

The main part of the proof of the Theorem 3 is the following:

If $0 \in \text{Int } (D(\varphi) - D(\psi))$ then (5) $(\varphi+\psi)^* = \varphi^* \nabla \psi^*$. This result improves the classical result of Moreau [6]. In other words, the condition $0 \in \text{Int}(D(\varphi)-D(\psi))$, which is in fact a Slater-stability condition, (weaker than the classical one), implies the equality (5) of the primal and dual problem, and the existence of a solution for the dual problem. More generally, in the variational situation studied by Ekeland-Temam, [5], one can prove the following statement:

Theorem (See for instance J. P. Aubin [1] Ch. 14).

Let V and Y two reflexive Banach spaces and

$A : V \to Y$ a continuous linear map

$\varphi : Y \to]-\infty,+\infty]$ a convex, ℓsc, proper function

$\psi : V \to]-\infty,+\infty]$ a convex, ℓsc, proper functions

such that $\mathrm{Int}\{AD(\psi) - D(\varphi)\} \ni 0$.

Then $\forall f \in V^*$ $\displaystyle\inf_{u\in V}\{\psi(u) + \varphi(Au) - (f,u)\} = - \min_{g\in Y^*}\{\psi^*(f-A^*g) + \varphi^*(g)\}$.

Taking $A = Id^+$ with $V = Y$ we get the previous result.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We prove that the topological condition Int $(D(A) - D(B)) \ni 0$, implies that the sum of the two maximal operators A and B is still maximal monotone.